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DETERMINATION OF THE FORM OF ATTACHMENT OF A ROD USING THE NATURAL FREQUENCIES OF ITS FLEXURAL OSCILLATIONS†

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The inverse problem of finding the form of the attachment of one of the ends of a rod, which is inaccessible to direct observation, from the natural frequencies of its flexural oscillations is considered. A theorem on the uniqueness of the solution of this inverse problem is proved and a method for establishing the unknown boundary conditions is indicated. An approximate formula for determining the boundary conditions is obtained using a finite set of natural frequencies (it is assumed that these natural frequencies can also be specified approximately with a certain degree of accuracy). The use of just the first non-zero natural frequencies is found to be essential here.

The problem in question belongs to the class of inverse problems and is a completely natural problem of acoustic diagnosis. However, it has not been formulated in this way (see [1–3]). A closely related formulation of the problem was proposed in [4]; namely, is it possible, knowing all the eigenvalues λ_n of the problem

$$\Delta U + \lambda_n U = 0, \quad x \in \Omega; \quad U|_{\partial\Omega} = 0$$

to determine the form of the domain Ω? Unlike this, in this paper it is not the form of the domain which is sought but the nature of the attachment.

Similarly formulated problems also arise in the spectral theory of differential operators, where it is required to establish the coefficients of a differential equation and the boundary conditions using a set of eigenvalues (for more details, see [5–8]). However, as data for finding the boundary conditions, it was not one spectrum (as in this paper) but several spectra or also other additional spectral data (for example, the spectral function, the Weyl function or so-called weighting numbers) that were used in those papers. Moreover, the main aim there was to determine the coefficients in the equation and not in the boundary conditions. The aim of this paper is, in the case of a known differential equation, to establish some of the boundary conditions of the eigenvalue problem from its spectrum.

The problem of determining a boundary condition using a finite set of eigenvalues has been considered previously [9]. Unlike that problem, in this paper it is necessary to determine not one but two boundary conditions, which requires other methods of solution.

1. FORMULATION OF THE INVERSE PROBLEM

The problem of the flexural oscillations of a rod with a rigidly clamped right-hand end reduces, after making the substitution $u(x, t) = y(x)\cos(\omega t)$ (see, for example, [10] or [11]) to the following eigenvalue problem

$$(\alpha y'')'' = \rho F\omega^2 y, \quad U_1(y,\omega) = 0, \quad U_2(y,\omega) = 0, \quad y(1) = 0, \quad y'(1) = 0 \tag{1.1}$$

Here, $U_i(y, \omega) = \sum_{j=1}^{4} a_{ij} y^{(j-1)}(0)$ $(i = 1, 2)$ are linear forms which characterize the fixing at the point $x = 0$ (at the left-hand end) and the coefficients $a_{ij} = a_{ij}(\omega)$ are polynomials of ω, α is the flexural rigidity, ρ is the density and F is the cross-section area of the rod.

We will now formulate the inverse of this eigenvalue problem: it is required to find the unknown linear forms $U_1(y, \omega)$, $U_2(y, \omega)$ from the natural frequencies of the oscillations of the rod.

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We will now reformulate the inverse problem in another, more precise, form, taking account of the fact that α, ρ and F are constant and denoting $\rho F\omega^2/\alpha$ by λ^4. In the new notation, problem (1.1) can be written as follows:

$$y^{(4)} = \lambda^4 y, \quad U_1(y, \lambda) = 0, \quad U_2(y, \lambda) = 0, \quad y(1) = 0, \quad y'(1) = 0 \tag{1.2}$$

where $U_i(y, \lambda) = \sum_{j=1}^{4} a_{ij} y^{(j-1)}(0)$ $(i = 1, 2)$ and the coefficients $a_{ij} = a_{ij}(\lambda)$ are polynomials in λ.

We shall denote the matrix, consisting of the coefficients a_{ij} of the forms $U_1(y, \lambda)$ and $U_2(y, \lambda)$, by A and its minors by M_{ij}:

$$A = \begin{Vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{Vmatrix}, \quad M_{ij} = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}$$

The search for the forms $U_1(y, \lambda)$, $U_2(y, \lambda)$ is equivalent to finding the linear envelope $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle$, constructed in the vectors $\mathbf{a}_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4})^T$ $(i = 1, 2)$.

The different cases for the clamping of one end of a rod [3, 10] are presented below:
rigid clamping

$$A = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}$$

free support

$$A = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Vmatrix}$$

free end

$$A = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

floating fixing

$$A = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

five other types of elastic fixing

$$A = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ c_1 & 0 & 0 & 1 \end{Vmatrix}, \quad \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -c_2 & 1 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 0 & 1 & 0 \\ c_1 & 0 & 0 & 1 \end{Vmatrix},$$
$$\begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & -c_2 & 1 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} c_1 & 0 & 0 & 1 \\ 0 & -c_2 & 1 & 0 \end{Vmatrix}$$

and a concentrated inertial element at the end

$$A = \begin{Vmatrix} -m\lambda^4 & 0 & 0 & 1 \\ 0 & -c\lambda^4 & 1 & 0 \end{Vmatrix}$$

Note that, in all ten cases

$$M_{14} \equiv 0, \quad M_{23} \equiv 0 \tag{1.3}$$

Hence, in terms of eigenvalue problem (1.2), the inverse problem which has been constructed above can be formulated as follows: the coefficients a_{ij} of the forms $U_1(y, \lambda)$ and $U_2(y, \lambda)$ of problem (1.2) are unknown, the rank of the matrix A, which is made up of these coefficients, is equal to two and the minors M_{14} and M_{23} of this matrix are identically equal to zero, the non-zero eigenvalues λ_k of problem (1.2) are known and it is required to find the linear envelope $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle$ of the vectors $\mathbf{a}_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4})^T$ $(i = 1, 2)$.

Conditions (1.3) do not constrain the physical formulation of the problem but they are necessary since, without them, the uniqueness of the solution of the inverse problem is destroyed (in this case, different linear envelopes may correspond to one and the same set of eigenvalues).

2. THE UNIQUENESS OF THE SOLUTION OF THE INVERSE PROBLEM

Together with problem (1.2), we consider the following eigenvalue problem

$$y^{(4)} = \lambda^4 y, \quad \tilde{U}_1(y, \lambda) = 0, \quad \tilde{U}_2(y, \lambda) = 0, \quad y(1) = 0, \quad y'(1) = 0 \tag{2.1}$$

where $\tilde{U}_i(y, \lambda) = \sum_{j=1}^{4} b_{ij} y^{(j-1)}(0)$ $(i = 1, 2)$ and $b_{ij} = b_{ij}(\lambda)$ are polynomials in λ.

We denote the matrix composed of the coefficients b_{ij} of the forms $\tilde{U}_1(y, \lambda)$ and $\tilde{U}_2(y, \lambda)$ by B and its minors by $\tilde{M}_{ij}$:

$$B = \begin{Vmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{Vmatrix}, \quad \tilde{M}_{ij} \equiv \begin{vmatrix} b_{1i} & b_{1j} \\ b_{2i} & b_{2j} \end{vmatrix}$$

The linear envelope of the vectors $\mathbf{b}_i = (b_{i1}, b_{i2}, b_{i3}, b_{i4})^T$ $(i = 1, 2)$ is denoted by $\langle \mathbf{b}_1, \mathbf{b}_2 \rangle$.

A theorem on the uniqueness of the solution of the inverse problem. Suppose the following conditions are satisfied

$$\operatorname{rank} A = \operatorname{rank} B = 2 \tag{2.2}$$

$$M_{14} \equiv M_{23} \equiv \tilde{M}_{14} \equiv \tilde{M}_{23} \equiv 0 \tag{2.3}$$

If the non-zero eigenvalues $\{\lambda_k\}$ of problem (1.2) and the non-zero eigenvalues $\{\tilde{\lambda}_k\}$ of problem (2.1) are identical, when account is taken of their multiplicities, then the linear envelopes $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle$ and $\langle \mathbf{b}_1, \mathbf{b}_2 \rangle$ are also identical.

Proof. We note that the functions

$$\begin{aligned} y_1(x, \lambda) &= (\cos \lambda x + \operatorname{ch} \lambda x)/2 \\ y_2(x, \lambda) &= (\sin \lambda x + \operatorname{sh} \lambda x)/(2\lambda) \\ y_3(x, \lambda) &= (-\cos \lambda x + \operatorname{ch} \lambda x)/(2\lambda^2) \\ y_4(x, \lambda) &= (-\sin \lambda x + \operatorname{sh} \lambda x)/(2\lambda^3) \end{aligned} \tag{2.4}$$

are linearly independent solutions of the equation

$$y^{(4)}(x, \lambda) = \lambda^4 y(x, \lambda) \tag{2.5}$$

which satisfy the conditions

$$y_j^{(r-1)}(0, \lambda) = \begin{cases} 0 & \text{when } j \neq r, \\ 1 & \text{when } j = r, \end{cases} \quad j, r = 1, 2, 3, 4 \tag{2.6}$$

(in other words, the solutions $y_j(x, \lambda)$ $(j = 1, 2, 3, 4)$ form a fundamental Cauchy system and are expressed in terms of Krylov functions [3]).

The following function

$$\Delta(\lambda) \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ y_1(1,\lambda) & y_2(1,\lambda) & y_3(1,\lambda) & y_4(1,\lambda) \\ y_1'(1,\lambda) & y_2'(1,\lambda) & y_3'(1,\lambda) & y_4'(1,\lambda) \end{vmatrix}$$

(conditions (2.6) have been taken into account here) is the characteristic determinant of the boundary-value problem when $\lambda \neq 0$.

Applying Laplace's theorem for evaluating determinants and using trigonometric formulae and equalities (2.4) as well as condition (2.3) of the theorem, we obtain

$$\Delta(\lambda) \equiv M_{12}\frac{\xi^-(\lambda)}{2\lambda^4} + M_{13}\frac{\eta^+(\lambda)}{2\lambda^3} + M_{24}\frac{\eta^-(\lambda)}{2\lambda} + M_{34}\frac{\xi^+(\lambda)}{2}$$

$$\xi^{\pm}(\lambda) = 1 \pm \cos\lambda \operatorname{ch}\lambda, \quad \eta^{\pm}(\lambda) = -\sin\lambda \operatorname{ch}\lambda \pm \cos\lambda \operatorname{sh}\lambda \tag{2.7}$$

The non-zero eigenvalues of problems (1.2) and (2.1) are the roots of the integral function $\Delta(\lambda)$ (see [12]).

In addition to roots which are identical to the non-zero eigenvalues of the problem, the characteristic determinant can also have a root $\lambda = 0$ of finite multiplicity.

Since $\Delta(\lambda) \not\equiv 0$, it follows from Hadamard's factorization theorem (see [13]) that the characteristic determinants $\Delta(\lambda)$ of problem (1.2) and $\tilde{\Delta}(\lambda)$ of problem (2.1) are connected by the relation

$$\Delta(\lambda) \equiv C\lambda^k e^{a\lambda}\tilde{\Delta}(\lambda)$$

where a is a certain real number, k is a certain non-negative integer and C is a certain non-zero constant. It follows from this that the right-hand side of relation (2.7) is identically equal to zero when M_{ij} is replaced by $M_{ij} - C\lambda^k e^{a\lambda}\tilde{M}_{ij}$ (Identity A).

Note that the number a in this identity is equal to zero. Actually, let us assume the opposite: $a \neq 0$. Then, the functions $\xi^{\pm}(\lambda)$ and $\eta^{\pm}(\lambda)$ and, also, the same functions multiplied by $e^{a\lambda}$ are polynomially independent. (We say that the functions $f_1(\lambda), f_2(\lambda), \ldots f_n(\lambda)$ are polynomially independent if their combination

$$P_1(\lambda)f_1(\lambda) + P_2(\lambda)f_2(\lambda) + \cdots + P_n(\lambda)f_n(\lambda)$$

with the arbitrary polynomials $P_1(\lambda), P_2(\lambda), \ldots P_n(\lambda)$ is identically equal to zero only in the case when $P_k(\lambda) \equiv 0$ $(k = 1, 2, \ldots, n)$.) The polynomial independence of these functions follows from the polynomial independence of the corresponding exponents. From this and from the Identity A, we obtain the identities

$$M_{12} \equiv \tilde{M}_{12} \equiv M_{13} \equiv \tilde{M}_{13} \equiv M_{24} \equiv \tilde{M}_{24} \equiv M_{34} \equiv \tilde{M}_{34} \equiv 0$$

which, in combination with identities (2.3), contradict condition (2.2) of the theorem.

Hence, $a = 0$. From this and from Identity A, by virtue of the polynomial independence of the corresponding functions, we obtain

$$(M_{12}, M_{13}, M_{14}, M_{23}, M_{24}, M_{34})^T \equiv C\lambda^k(\tilde{M}_{12}, \tilde{M}_{13}, \tilde{M}_{14}, \tilde{M}_{23}, \tilde{M}_{24}, \tilde{M}_{34})^T$$

which is equivalent to the proportionality of the bivectors $\mathbf{a}_1 \wedge \mathbf{a}_2$ and $\mathbf{b}_1 \wedge \mathbf{b}_2$.

It is well-known [14] that there is a natural bijective correspondence between the classes of non-zero, proportional bivectors and the two-dimensional subspaces of a vector space. In this correspondence, a vector product $\mathbf{x}_1 \wedge \mathbf{x}_2$ of the vectors of its arbitrary basis $\mathbf{x}_1, \mathbf{x}_2$ corresponds to each subspace and a subspace $\mathbf{x}_1 \wedge \mathbf{x}_2$ corresponds to each bivector $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle$. It therefore follows from the last identity that $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = \langle \mathbf{b}_1, \mathbf{b}_2 \rangle$ which it was required to prove.

Remark 1. As was noted above, conditions (1.3) do not constrain the physical formulation of the problem but are essential for the uniqueness of finding the boundary conditions. Actually, the boundary conditions $y(0) = 0$, $y'''(0) = 0$ $(M_{14} = 1 \neq 0)$ and $y'(0) = 0, y''(0) = 0$ $(M_{23} = 1 \neq 0)$ are not equivalent but the characteristic determinants of the corresponding eigenvalue problems (1.2) and (2.1) are identical. It follows from this that the abstract boundary conditions are uniquely defined using the non-zero eigenvalues of the corresponding eigenvalue problem.

3. AN ALGORITHM FOR DETERMINING THE BOUNDARY CONDITIONS FROM THE NATURAL FREQUENCIES

It has been shown above that the problem of finding the unknown linear forms $U_1(y, \lambda)$ and $U_2(y, \lambda)$ from the natural frequencies of flexural oscillations of a rod has a unique solution (in the sense that the linear envelopes, composed of the coefficients of these linear forms, are uniquely defined). The next question is how can this solution be constructed?

Further, since instruments for measuring the natural frequencies (spectrometers) cannot record the infinite set of frequencies of a system and, furthermore, small errors are possible when measuring the natural frequencies, the problem also arises of finding an algorithm for the approximate determination of the type of fixing of a rod from a finite set of the first natural frequencies which have been found with a certain error.

This section is concerned with solving these problems and constructing exact and approximate solutions.

The algorithm for determining the linear forms of the boundary conditions is explained in the case of the natural physical assumptions that the coefficients c_1 and c_2 of the matrix A are non-negative. (In this case, the solution is simplified in view of the fact that $\lambda = 0$ is not an eigenvalue)

Since the linear envelope $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle$ is found from all the second-order minors of the matrix A using well-known methods of linear algebra [14], it remains to explain how the unknown minors

$$M_{12}, M_{13}, M_{24}, M_{34} \tag{3.1}$$

are obtained.

It follows from representation (2.7) that $\Delta(\lambda)$ is an even, first-order, integral function and, if the value $\lambda_j > 0$ is a root of this equation, then the values $-\lambda_j$, $-i\lambda_j$, $i\lambda_j$ are also roots of this equation.

It follows from this and from Hadamard's factorization theorem [13] that the characteristic determinant $\Delta(\lambda)$ of problem (1.2) admits of the representation

$$\Delta(\lambda) \equiv K \prod_{j=1}^{\infty} \left(1 - \frac{\lambda^4}{\lambda_j^4}\right) \tag{3.2}$$

where K is an arbitrary, non-zero constant and λ_j are the positive eigenvalues of problem (1.2).

Hence, if all of the positive eigenvalues λ_j of problem (1.2) are known, the identity in λ follows from representations (2.7) and (3.2).

The idea behind the method for the approximate determination of the form of the boundary conditions of the problem of the oscillations of a rod from the first s eigenvalues μ_j consists of replacing the infinite product on the right-hand side of this identity by a finite product. In this case, the eigenvalues μ_j may only approximate to the true eigenvalues λ_j.

Hence, instead of the identity being considered, we can write the approximate identity

$$M_{12} \frac{\xi^-(\lambda)}{2\lambda^4} + M_{13} \frac{\eta^+(\lambda)}{2\lambda^3} + M_{24} \frac{\eta^-(\lambda)}{2\lambda} + M_{34} \frac{\xi^+(\lambda)}{2} \approx K \prod_{j=1}^{s} \left(1 - \frac{\lambda^4}{\mu_j^4}\right) \tag{3.3}$$

The minors (3.1) can be found, using different methods, from relation (3.3) (or from the exact identity (3.2), (2.7) if the exact solution is required), apart from a coefficient. For example, they can be found using the system of functions which is conjugately biorthogonal to the system of functions

$$\frac{\xi^-(\lambda)}{2\lambda^4}, \quad \frac{\eta^+(\lambda)}{2\lambda^3}, \quad \frac{\eta^-(\lambda)}{2\lambda}, \quad \frac{\xi^+(\lambda)}{2}$$

in the space L_2.

However, two methods, based on the derivation of a system of linear algebraic equations from the corresponding identity, turn out to be simpler to implement.

The first of these demonstrates how exact values of the minors (3.1) can be found from the whole infinite set of natural frequencies. It is based on the derivation of a system of linear algebraic equations with a well-posed matrix from the the exact identity (3.2), (2.7) (for the concept of an ill-posed matrix and a well-posed matrix, see [15]). The second method was used to find an approximate solution and is based on the derivation from (3.3) of an indeterminate system of three linear algebraic equations in four unknowns.

First method. On assigning four different specific values z_k (k = 1, 2, 3, 4) to the parameter λ, we obtain a system of four linear algebraic equations in the four unknowns (3.1). This system may turn out to be indeterminate or ill-posed. This situation arises, in particular, if the eigenvalues λ_k are chosen as the values z_k (a corresponding example is considered below in Remark 2).

However, the numbers z_k (k = 1, 2, 3, 4) can be chosen such that the corresponding matrix of the system will be determinate and well-posed.

The best set of numbers z_k (k = 1, 2, 3, 4) we found was

$$z_1 = 7.854757438, \quad z_2 = 5.82239, \quad z_3 = 3.98955, \quad z_4 = 3.72621 \tag{3.4}$$

On assigning the values (3.4) to the parameter λ, we obtain a system of four linear algebraic equations in the four unknowns (3.1)

$$M_{12}\frac{\xi^-(z_k)}{2z_k^4}+M_{13}\frac{\eta^+(z_k)}{2z_k^3}+M_{24}\frac{\eta^-(z_k)}{2z_k}+M_{34}\frac{\xi^+(z_k)}{2}=K\prod_{j=1}^{\infty}\left(1-\frac{z_k^4}{\lambda_j^4}\right) \tag{3.5}$$

where K is an arbitrary, non-zero constant. (To be specific, the constant K can be chosen to be equal to any number.)

The minors (3.1) are found from this system using well-known methods.

The same method can be used for the approximate determination of the unknowns (3.1) using the first s values of $\mu_j \approx \lambda_j$ by replacing the infinite product by a finite product.

However, this method is found to be ineffective for a numerical solution. We checked different forms of attachment of the left end of a rod using the MAPLE software package. Generalizing the data obtained, it can be said that, for all modes of attachment of the rod in the case of the set of numbers z_k (k = 1, 2, 3, 4) in the form (3.4), if the magnitude of s from (3.3) is equal to 100 and the accuracy of the chosen values μ_j is $\varepsilon = 10^{-7}$, then the bivector **M** is found with a much reduced accuracy (in certain cases, it is found with an accuracy of 10^{-3}). If, however, s = 50, the accuracy of the calculation of each component of the bivector can be 10^{-2}, that is, it deteriorates by five orders of magnitude. Hence, in order to find an approximate solution using the first method, it is necessary to know an enormous number of eigenvalues of the problem, which have been found with a sufficiently high degree of accuracy. This is unacceptable in practice. The second method is more efficient.

Second method. We substitute the values μ_j (j = 1, 2, 3) which are approximately the same as the first three positive eigenvalues of problem (1.2), into (3.3). We obtain a system of three homogeneous algebraic equations in the four unknowns (3.1)

$$M_{12}\frac{\xi^-(\mu_j)}{2\mu_j^4}+M_{13}\frac{\eta^+(\mu_j)}{2\mu_j^3}+M_{24}\frac{\eta^-(\mu_j)}{2\mu_j}+M_{34}\frac{\xi^+(\mu_j)}{2}\approx 0 \tag{3.6}$$

The resulting system has an infinite set of solutions. It follows from the uniqueness theorem which has been proved and from approximate identity (3.3) (where the magnitude of s is chosen to be equal to three), that the unknown minors (3.1) can be found approximately apart from a coefficient. Hence, if μ_j (j = 1, 2, 3) differ insignificantly from the first three eigenvalues, then the resulting system must have a rank of 3 and a solution, determined apart from a constant. Calculations, carried out using the MAPLE software package confirmed this. The unknown minors are found, apart from a constant. In this case, the order of the error in the calculations is frequently hardly different from the error in the closeness of the values of μ_j and λ_j and only in certain cases can it deteriorate by four orders of magnitude. The principal advantage of this method lies in the fact that its application only requires the use of the first three natural frequencies.

Example. If μ_1 = 4.7300407, μ_2 = 7.8532046, μ_3 = 10.9956078 are the values of $(\rho F\omega_i^2/\alpha^{1/2})$ corresponding to the first three natural frequencies ω_i determined using a spectrometer, then the solution of system (3.6), apart from a constant, has the form

$$M_{12}=1, \quad M_{13}=0.91\times10^{-8}, \quad M_{24}=-0.94\times10^{-10}, \quad M_{34}=-0.13\times10^{-7} \tag{3.7}$$

Moreover, according to a condition of the problem $M_{14} = 0, M_{23} = 0$.

We now find the linear envelope corresponding to these minors. Suppose $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$ is an arbitrary vector of the required linear envelope $\langle \mathbf{a}_1, \mathbf{a}_2\rangle$. Then, the coordinates of the vector $\mathbf{x}$ satisfy the condition

$$\operatorname{rank}\begin{Vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ x_1 & x_2 & x_3 & x_4 \end{Vmatrix} = 2 \tag{3.8}$$

Since

$$M_{12} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

condition (3.8) is equivalent to both of the bordering M_{12} minors vanishing.

On expanding the corresponding determinants with respect to the third row, we obtain

$$x_1 M_{23} - x_2 M_{13} + x_3 M_{12} = 0, \quad x_1 M_{24} - x_2 M_{14} + x_4 M_{12} = 0$$

We now substitute the values of M_{ij} from (3.7) into these equalities. The values of M_{13} and M_{24} can be assumed to be equal to zero (the degree of accuracy is equal to 10^{-7}, that is, quite high). It can therefore be assumed that $x_3 = 0$ and $x_4 = 0$, and the arbitrary vector of the linear envelope has the form $\mathbf{x} = (x_1, x_2, 0, 0)^T$.

As the basis vectors of this linear envelope, one can choose, for example, the vectors $\mathbf{a}_1 = (1, 0, 0, 0)^T$ and $\mathbf{a}_2 = (1, 0, 0, 0)^T$

Consequently, the required boundary conditions have the form

$$y(0) = 0, \quad y'(0) = 0$$

This means that the left end of the rod, which is inaccessible for direct observation, is rigidly clamped.

Note that the fixing of the end of the rod has been correctly determined. The numbers μ_1, μ_2, μ_3 presented above are almost the same as the first three roots of the equation

$$1 - \cos\lambda \operatorname{ch}\lambda = 0$$

The accuracy of the approximation is equal to 10^{-7}.

Remark 2. To what extent is the use of only the first non-zero eigenvalues essential for the approximate establishment of the boundary conditions? Is it sufficient to use an arbitrary finite set of eigenvalues (natural frequencies) in order to determine the boundary conditions? We will now present an example which shows that even the use of an infinite set of non-zero eigenvalues as the data still does not guarantee uniqueness in establishing the boundary conditions.

The eigenvalues of the problem

$$y^{(4)} = \lambda^4 y, \quad y(0) = 0, \quad y'(0) = 0, \quad y(1) = 0, \quad y'(1) = 0 \tag{3.9}$$

starting from the seventh (see [3, 16]), are almost identical with the numbers $\lambda_k = (k + 1/2)\pi$, while the eigenvalues of the problem

$$y^{(4)} = \lambda^4 y, \quad y''(0) = 0, \quad y'''(0) = 0, \quad y(1) = 0, \quad y'(1) = 0 \tag{3.10}$$

starting from the seventh, are almost identical with the numbers $\bar{\lambda}_k = (k - 1/2)\pi$.

Consequently, the sets $\{\lambda_{k-1}\}$ and $\{\bar{\lambda}_k\}$ are practically identical in the infinite set of numbers $\{(k + 1/2)\pi\}$ ($k = 8, 9, \ldots$). However, the corresponding boundary conditions are very different.

Calculations, carried out using the MAPLE software package, lead to the same results. If, instead of μ_j, the hundredth, hundredth and first and hundredth and second roots of the equation

$$1 + \cos\mu \operatorname{ch}\mu = 0$$

which have been found with an accuracy of 10^{-7}, are substituted into system (3.6), the required minors, calculated using a computer, apart from a constant have the form

$$M_{12} = 1, \quad M_{13} = -0.58\times10^{-8}, \quad M_{24} = 0.53\times10^{-13}, \quad M_{34} = -0.95\times10^{-10}$$

These minors do not correspond to the required boundary conditions of problem (3.10) but to the boundary conditions of problem (3.9).

Hence, it is precisely the first non-zero eigenvalues that are essential for uniqueness in establishing the boundary conditions.

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REFERENCES

1. DENISOV, A. M., *Introduction to the Theory of Inverse Problems*. Izd. Mosk. Gos. Univ., Moscow, 1994.
2. ARTOBOLEVSKII, I. I., BOBROVNITSKII, Yu. I. and GENKIN, M. D., *Introduction to the Acoustic Dynamics of Machines*. Nauka, Moscow, 1979.
3. BOLOTIN, V. V. (Ed.), *Vibrations in Engineering: A Handbook*, Vol. 1, *Oscillations of Linear Systems*. Mashinostroyeniye, Moscow, 1978.
4. KAC, M., Can one hear the shape of a drum? *A. Math. Monthly*, 1966, **73**, 4, 1–23.
5. BORG, G., EineUmkehrung der Sturm–Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte. *Acta Math*., 1946, **78**, 1–96.
6. LEVITAN, B. M., *Inverse Sturm–Liouville Problems*. Nauka, Moscow, 1984.
7. SADOVNICHII, V. A., Uniqueness of the solution of an inverse problem for a second order equation with non-decomposing boundary conditions. *Vest. Mosk. Gos. Univ. Ser. 1. Matematika, Mekhanika*, 1974, 1, 143–151.
8. YURKO, V. A., The *Inverse Problem for Differential Operators*, Izd. Saratov Univ., Saratov, 1989.
9. AKHTYAMOV, A. M., The determination of the boundary condition from a finite set of eigenvalues. *Differents. Uravneniya*, 1999, **35**, 8, 1127–1128.
10. COLLATZ, K., *Eigenwertaufgaben mit technischen Anwendungen*. Akad. Verlagsgesellschaft Geest and Porting K.-G, Leipzig, 1963.
11. STRUTT, W. (Lord Rayleigh) V., *The Theory of Sound*. V. I. L.: Macmillan, 1929.
12. NAIMARK, M. A., *Linear Differential Operators*. Nauka, Moscow, 1969.
13. LEVIN, B. Ya., *Distribution of the Roots of Integral Functions*. Gostekhizdat, Moscow, 1956.
14. POSTNIKOV, M. M., *Linear Algebra and Differential Geometry*. Nauka, Moscow, 1979.
15. FADDEYEV, D. K. and FADDEYEVA, V. N., *Computational Methods of Linear Algebra*. Fizmatgiz, Moscow, 1963.
16. TIMOSHENKO, S., *Vibration Problems in Engineering*. D. Van Nostrand Company, Toronto, 1955.

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